

Planar β -skeletons via point location in monotone subdivisions of subset of lunes *

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Abstract

We present a new algorithm for lune-based β -skeletons for sets of n points in the plane, for $\beta \in (2, \infty]$, the only case when optimal algorithms are not known. The running time of the algorithm is $O(n^{3/2} \log^{1/2} n)$, which is the best known and is an improvement of Rao and Mukhopadhyay [10] result. The method is based on point location in monotonic subdivisions of arrangements of curve segments.

1 Introduction

β -skeletons [7] belong to the family of proximity graphs, geometric graphs in which two vertices (points) produce an edge if and only if they satisfy particular geometric requirements. The proximity graphs are both important and popular because of many practical applications. They span a broad spectrum of areas, from earlier work on mathematical morphology on lattices and graphs, pattern recognition, geographic information systems, data mining to more recent applications in ad hoc wireless networks and systems biology. Requirements defining proximity graphs can be formulated in many metrics, with the Euclidean metric being one of the most commonly used.

Definition 1.1 *For a given set P containing n points in R^2 , distance function d and the parameter β we define β -skeleton as a graph (P, E) , in which $xy \in E$ iff no point in $P \setminus \{x, y\}$ belongs to $R(x, y, \beta)$, where*

- for $\beta = 0$, $R(x, y, \beta)$ is the segment xy ;
- for $0 < \beta < 1$, $R(x, y, \beta)$ is the intersection of two discs with the radius $d(x, y)/2\beta$, which boundaries contain the both points x and y .

For $1 \leq \beta \leq \infty$ there are two ways to define the region $R(x, y, \beta)$ leading to two different families of graphs.

The lune-based definition is as follows:

- For $1 \leq \beta \leq \infty$, $R(x, y, \beta)$ is a intersection of two discs with radius $\beta d(x, y)/2$ and centered in points $(1 - \beta/2)x + (\beta/2)y$ and $(\beta/2)x + (1 - \beta/2)y$, respectively.

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- For $\beta = \infty$, $R(x, y, \beta)$ is an unbounded strip between two lines containing x and y , respectively, and perpendicular to the segment xy .

The second definition defining a different family of graphs is circle-based.

- For $1 \leq \beta < \infty$, $R(x, y, \beta)$ is the union of two discs, each having diameter $\beta d(x, y)$ and having the segment xy as its chord.
- For $\beta = \infty$, $R(x, y, \beta)$ is the union of hyperplanes having the segment xy on their border.

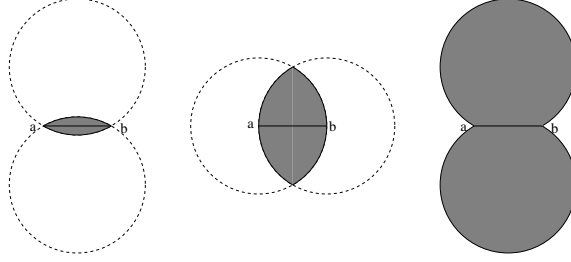


Figure 1: The region $R(a, b, \beta)$ for $0 < \beta < 1$ (left), the lune-based region $R(a, b, \beta)$ for $1 \leq \beta < \infty$ (middle) and the circle-based region $R(a, b, \beta)$ for $1 \leq \beta < \infty$ (right).

The open (closed, respectively) area $R(x, y, \beta)$ defines an open (closed, respectively) β -skeleton. A Gabriel Graph (GG) [4] is a closed 1-skeleton and a Relative Neighborhood Graph (RNG) [12] is an open 2-skeleton.

For $0 \leq \beta < 1$, there are in the worst case optimal $O(n^2)$ time algorithms [5] for both lune-based and circle-based β -skeletons. Also, any β -skeleton for $1 \leq \beta \leq 2$ can be computed in the optimal $O(n \log n)$ time ([6, 8, 9, 11]). For circle-based β -skeletons, where $1 \leq \beta \leq \infty$, there also is an optimal $O(n \log n)$ time algorithm [7].

For $\beta > 2$, the lune-based and circle-based regions $R(x, y, \beta)$ are very different and it has been reflected by higher running times of the best known algorithms for the lune-based β -skeletons. In particular, for $2 < \beta \leq \infty$, the fastest algorithm that has been reported for computing lune-based β -skeletons requires $O(n^{3/2} \log n)$ time [10]. The algorithm uses the sweeping-line technique. In this paper, we will show a $O(n^{3/2} \log^{1/2} n)$ time algorithm computing a lune-based β -skeleton for $2 < \beta \leq \infty$. To this end, we will use arrangements of curve segments [1] and data structure for point location in monotone subdivisions [3].

2 Algorithm

It is well known, see e.g. [7], that β -skeletons for $2 < \beta$ are subgraphs of the Delaunay triangulation of the input point sets. For a set P of n points, they have size $O(n)$. Thus, to construct a β -skeleton, we can use their definition to eliminate edges that do not belong to β -skeleton by comparing edges of Delaunay triangulation against all of the points in the input set. To this end, we analyze lunes (for $2 < \beta < \infty$) or strips (for $\beta = \infty$) defined by the edges of the DT triangulation.

Specifically, we have to identify these lunes or strips that contain points in P . Let us consider m edges of the Delaunay triangulation of P . m will be determined later to minimize the running time.

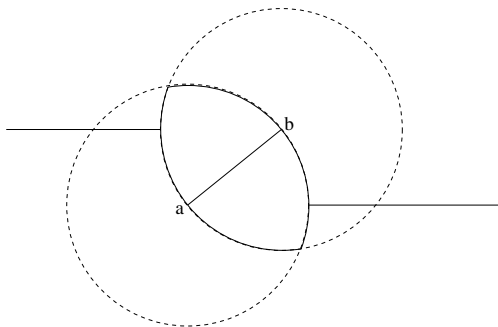


Figure 2: A subdivision of the plane into three monotone regions for one lune.

We construct arrangement of lunes defined by these edges. Specifically, for each of the lunes, we draw two horizontal half-lines that start at the points with the smallest and, respectively, largest x -coordinates. They yield three monotone regions per lune (see Fig. 2). The size of the arrangement is $O(m^2)$ and the time complexity of its construction is the same [1].

The intersection of monotone regions is monotone. The above construction results in a monotone subdivision of size $O(m^2)$ for the arrangement of size $O(m^2)$.

Then, we locate points of P in regions of the subdivision. The construction time of an auxiliary data structure is linear with respect to a size of the subdivision [3]. Hence it can be done in $O(m^2)$ time and $O(m^2)$ space. A query time is $O(\log m)$. The results of querying are stored in respective cells of the subdivision data structure.

We traverse the dual graph of the subdivision in a Depth-First-Search fashion in order to identify (and store) edges of the Delaunay triangulations that do not belong the β -skeleton and lunes (strips, respectively) containing currently visited region of the subdivision.

The structure is a table of lunes (strips, respectively) forming the subdivision with

- records storing - value *true* when the respective lune contains some point of P and *false* otherwise and
- pointers to a double linked list of lunes (strips, respectively) containing currently visited region which respective records are *false*.

Moving between two neighboring regions we visit or leave the lune or our status stays unchanged. In the first case :

- if record of the auxiliary structure of this lune has value *false* we add the lune to the list in the structure; otherwise the list stays unchanged,
- if visited region contains a point of P we change values of records of all lunes belonging to the list on *true* and erase the list.

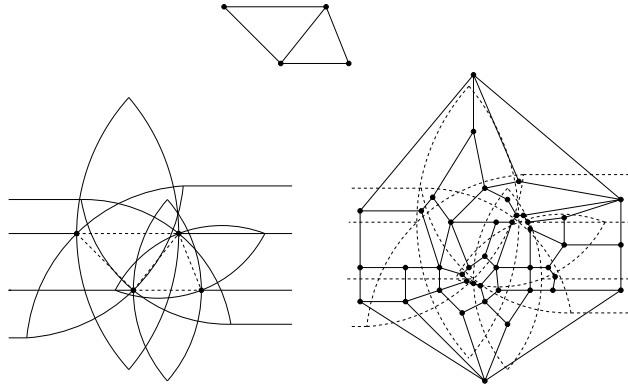


Figure 3: The Delaunay triangulation for a set of four points, the corresponding subdivision and its dual graph.

In the second case we remove the lune from the list if its record has value *false*.

In the third case all data structures stay unchanged.

Visiting all regions of the subdivision take $O(m^2)$ time.

We repeat the above process for all the groups in the partition of the edges into groups of size m .

Theorem 2.1 *For $2 < \beta \leq \infty$, β -skeleton of a set P of n points in the plane can be constructed in $O(n^{3/2} \log^{1/2} n)$ time and $O(n \log n)$ space.*

PROOF: The time complexity of every steps of the algorithm is $O(n \log n) + n/m(O(m^2) + O(m^2) + O(n \log m) + O(m^2)) = O(n \log n) + n/m(O(m^2) + O(n \log m))$. The value of this function is minimal for $m = (n \log n)^{1/2}$. Hence the time complexity of the algorithm is $O(n^{3/2} \log^{1/2} n)$. Space complexity of the algorithm is $O(n) + O(m^2) = O(n \log n)$. \square

3 Conclusion

We have presented a new algorithm for the lune based β -skeletons, where $\beta > 2$. For circle-based skeletons the approach would work as well, however better algorithms are known in that case. Although the algorithm provides only an incremental improvement compared to the previous one, the improvement is interesting due to a tantalizing jump in the algorithmic complexity of the β -skeleton problem that occurs for $\beta = 2$. While for $\beta \leq 2$ optimal algorithms are known, the optimality is open for $\beta > 2$. The presented algorithm narrows the gap, however more research is needed to better understand the reasons for this apparent difficulty for $\beta > 2$.

Finally, we believe that the point location approach can be also used for metrics other than Euclidean.

References

- [1] N. M. Amato, M. T. Goodrich, E. A. Ramos. *Computing the arrangement of curve segments: Divide-and-conquer algorithms via sampling*. Proc. 11th ACM-SIAM Sympos. Discrete Algorithms, 2000, 705-706.
- [2] J. Cardinal, S. Collette, S. Langerman. *Empty region graphs*. Comput. Geom. 42(3), 2009, pp. 183-195.
- [3] H. Edelsbrunner, L. J. Guibas, J. Stolfi. *Optimal point location in a monotone subdivision*. SIAM Journal on Computing, 15(2):317-340, 1986.
- [4] K.R. Gabriel, R.R. Sokal. *A new statistical approach to geographic variation analysis*. Systematic Zoology 18 (1969) 259-278.
- [5] F. Hurtado, G. Liotta, H. Meijer. *Optimal and suboptimal robust algorithms for proximity graphs*. Computational Geometry 25 (2003), pp. 35-49.
- [6] J.W. Jaromczyk, M. Kowaluk, F. Yao. *An optimal algorithm for constructing β -skeletons in l_p metric*. Manuscript, 1989.
- [7] D.G. Kirkpatrick, J.D. Radke. *A framework for computational morphology*. in: G.T. Toussaint (Ed.), Computational Geometry, North-Holland, Amsterdam, 1985, pp. 217-248.
- [8] A. Lingas. *A linear-time construction of the relative neighborhood graph from the Delaunay triangulation*. Computational Geometry 4 (1994) 199-208.
- [9] D.W. Matula, R.R. Sokal. *Properties of Gabriel graphs relevant to geographic variation research and clustering of points in the plane*. Geogr. Anal. 12 (3) (1980) 205-222.
- [10] S.V. Rao, A. Mukhopadhyay. *Fast algorithm for computing β -skeletons and their relatives*. Proc. 8th Annu. Int. Symp. on Algorithms and Computation, in: Lecture Notes Comput. Sci., Vol. 1350, 1997, pp. 374-383.
- [11] K.J. Supowit. *The relative neighborhood graph with an application to minimum spanning trees*. J. ACM 30 (3) (1983) 428-448.
- [12] G. Toussaint. *The relative neighborhood graph of a finite planar set*. Pattern Recognition 12 (1980), 261-268.